

Regularity in algebraic frames

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Abstract

This article considers algebraic frames in which the meet of two compact elements is compact, and, in that context, when the subframe of all regular elements is itself regular. Motivated by the study of a frame of convex ℓ -subgroups of a lattice-ordered group, a number of relevant sufficient conditions are given for this subframe to be regular. An example is given of a frame of convex ℓ -subgroups for which the subframe of regular elements is not regular.

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The investigation of [22] characterized the regular algebraic frames, and also when certain distinguished elements of an algebraic frame are regular; this is the content of [22, Theorem 2.4]. The discussion of [20] turns upon the introduction of two sublattices of an algebraic frame L : the subframe (resp. complete sublattice) $\text{FP}(L)$ (resp. $\text{CP}(L)$) of L generated by the set of polars PL ; it is demonstrated there that the two are different. Among the distinguished sublattices of $\text{FP}(L)$ that arise, is the one comprised of the “pure” elements—the term abstracted from the notion of a pure ideal of a commutative ring with identity [9]. In the sequel it is shown that this notion coincides with that of a regular element (Section 3). This equivalence then leads one to reconsider the situation of [22]. This article should then be viewed as a successor to [22] and [20].

In this article the discourse centers on algebraic frames in which the meet of two compact elements is compact. The objective is to study $\text{Reg}(L)$, the subframe of regular elements of the frame L , and, in particular, the question of when $\text{Reg}(L)$ coincides with the regular coreflection of L . When the relation \leq , the so-called “well below” relation in frames, interpolates, it is rather easy to conclude that $\text{Reg}(L)$ is regular, and, therefore, the regular coreflection. In this article, we look for conditions which imply that \leq interpolates.

It is well known that if the frame is normal then \leq interpolates. We shall leave the general question of when an algebraic frame is normal for another exposition [21]. The general discussion below concerning $\text{Reg}(L)$ is cast in terms of a description of the induced map from the spectrum of L to that of $\text{Reg}(L)$. The best result to date in this regard is Theorem 7.1; it leads to an example of an algebraic frame with the FIP and disjointification, such that the frame of regular elements is not regular.

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We begin with a review of the frame theory we shall require. The reader who is knowledgeable about frames in general will probably be able to skip lightly through most of the first section.

1. Frame-theoretic preliminaries

The commentary below is a catalogue of background material on frames and algebraic frames, in particular. We refer the reader to [15,16,26] for general background on frames, and to [22] for additional material on closure operators.

Definition & Remark 1.1. Throughout, L is a complete lattice. The top and bottom are denoted 1 and 0, respectively. For $x \in L$, $\uparrow x$ (resp. $\downarrow x$) stands for the set of elements $\geq x$ (resp. $\leq x$). Let us also point out to the reader that, throughout, we use the phrase “ y exceeds x ” in a poset to indicate that $y \geq x$.

1. $c \in L$ is *compact* if $c \leq \bigvee_{i \in I} x_i$ implies that $c \leq \bigvee_{i \in F} x_i$, for a suitable finite subset F of I . L is *algebraic* if each $x \in L$ is a supremum of compact elements. $\mathfrak{k}(L)$ stands for the set of compact elements of L . If 1 is compact it is said that L is *compact*.
2. L is said to have the *finite intersection property* (always abbreviated FIP) if for any pair $a, b \in \mathfrak{k}(L)$ it follows that $a \wedge b \in \mathfrak{k}(L)$. Observe that $\mathfrak{k}(L)$ is always closed under taking finite suprema. L is *coherent* if it is compact and has the FIP.
3. L is a *frame* if the following distributive law holds for each $S \subseteq L$:

$$a \wedge \left(\bigvee S \right) = \bigvee \{a \wedge s : s \in S\}.$$

It is well known that an algebraic lattice is a frame as long as it is distributive.

4. $p \in L$ is *prime* if $p < 1$ and $x \wedge y \leq p$ implies that $x \leq p$ or $y \leq p$. $\text{Spec}(L)$ shall denote the set of prime elements of L . We think of $\text{Spec}(L)$ as endowed with the *hull-kernel* topology. To recall, this is the topology whose open sets are of the form

$$c(x) = \{p \in \text{Spec}(L) : x \not\leq p\} \quad (x \in L).$$

5. Let L be a frame; a^\perp denotes the supremum of all x in L for which $a \wedge x = 0$. Call $p \in L$ a *polar* if it is of the form $p = y^\perp$, for some $y \in L$. It is well known that the set PL of all polars forms a complete boolean algebra, in which infima agree with those in L .
6. Let L be a frame. Recall that $a \leq b$ if $b \vee a^\perp = 1$; if $a \leq b$ one says that a is *well below* b . $x \in L$ is *regular* if

$$x = \bigvee \{a \in L : a \leq x\}.$$

L is *regular* if each element of L is regular. Let $\text{Reg}(L)$ denote the subset of all regular elements of L . It is well known that $\text{Reg}(L)$ is a subframe of L .

This should not be confused with the *regular coreflection* of L , which is the join of all the regular subframes of L . The latter, which we will denote $\varrho(L)$, is the largest regular subframe of L , and, evidently, $\varrho(L) \subseteq \text{Reg}(L)$. In general, $\text{Reg}(L)$ fails to be regular, and $\text{Reg}(\cdot)$ has to be iterated to attain $\varrho(L)$. Example 7.2 exhibits an algebraic frame with the FIP for which $\text{Reg}(L)$ is the three-element frame $\{0, a, 1\}$, while $\varrho(L) = \{0, 1\}$.

In Section 4 we take up the discussion of when $\text{Reg}(L)$ and $\varrho(L)$ coincide.

We shall say that \leq *interpolates* if $x \leq y$ implies that there exists a $z \in L$ such that $x \leq z \leq y$. It is well known that, in general, \leq does not interpolate.

7. The frame L is said to be *normal* if $1 = x \vee y$ implies that there exist disjoint u and v (which may be taken $u \leq x$ and $v \leq y$, respectively), such that $u \vee v = 1 = x \vee y$. We note – see [26] – that, in any normal frame, \leq does interpolate.
8. [16]. Let L be a frame and suppose that $j : L \rightarrow L$ is a closure operator; jL designates $\{x \in L : j(x) = x\}$. j is a *nucleus* if $j(a \wedge b) = j(a) \wedge j(b)$. We say that jL is *nuclear* when j is a nucleus.
9. [22, Section 4]. Suppose that L is an algebraic lattice, and j is a closure operator. Say that j is *inductive* if

$$j(x) = \bigvee \{j(a) : a \in \mathfrak{k}(L), a \leq x\}.$$

Then jL is algebraic and $\mathfrak{k}(jL) = j(\mathfrak{k}(L))$. If L is also a frame and j is a nucleus on L , then jL is an algebraic frame as well; its members are called *j -elements*. Observe, in addition, that if L is an algebraic frame and j is an inductive nucleus on L , then

- (a) $\text{Spec}(jL) = \text{Spec}(L) \cap jL$;
 (b) if L has the FIP then so does jL .
10. [22, Section 4]. Suppose that L is an algebraic frame with the FIP and that j is a nucleus on L . Let $\text{Ab}(j)$ stand for the set of all $x \in L$ such that $a \leq x$ (with a compact) implies that $j(a) \leq x$. Then $\text{Ab}(j)$ is an algebraic frame with the FIP. More precisely,

$$\widehat{j}(x) = \bigvee \{j(a) : a \in \mathfrak{k}(L), a \leq x\}$$

defines an inductive nucleus such that $\widehat{j}L = \text{Ab}(j)$.

11. Closure operators on L are partially ordered by defining $j_1 \leq j_2$ if $j_1(x) \leq j_2(x)$ for each $x \in L$, which, in turn, is equivalent to $j_2L \subseteq j_1L$. Under these stipulations, and using the notation of 10, \widehat{j} is the largest inductive closure operator below j . The passage $j \mapsto \widehat{j}$ is referred to as *inductivization*.

Escardó (in [10]) considers inductivization in a more general context. What he terms a finitary nucleus is exactly the concept of an inductive nucleus on an algebraic frame.

12. The nucleus j is *dense* if $j(0) = 0$. Note that j is dense if and only if $0 \in jL$.

Remark 1.2. It is worth underscoring that we shall assume and liberally apply Zorn's Lemma, which guarantees that all algebraic frames are spatial.

We recall some basic information on algebraic frames. The first part of the following commentary actually applies to arbitrary frames.

Definition & Remark 1.3. (a) Suppose that L is a frame. In general, one has the following result—see [4, p. 130]: the following are equivalent.

1. Each polar is complemented.
2. For each $a, b \in L$, $a^\perp \vee b^\perp = (a \wedge b)^\perp$.
3. PL is a sublattice of L .

(b) Suppose now that L is an algebraic frame with the FIP. $\text{Min}(L)$ denotes the set of all minimal primes of L . It is well known that each $p \in \text{Min}(L)$ is a join of polars of the form c^\perp , with $c \not\leq p$, and $c \in \mathfrak{k}(L)$; this is essentially the content of [22, Lemma 2.2]. Note as well that each polar is an infimum of minimal primes.

Moreover, recall that $x \in L$ is a *d-element* if $c \leq x$, with $c \in \mathfrak{k}(L)$, implies that $c^{\perp\perp} \leq x$. (See [22, Section 5] for a detailed discussion of *d-elements*. The concept is motivated by the *d-ideals* of [13,14].) We have the associated inductive nucleus d on L defined by $d = \widehat{(\cdot)^{\perp\perp}}$. Note as well, in the terminology of 1.1.10, that $dL = \text{Ab}((\cdot)^{\perp\perp})$.

Let Min^* stand for the nucleus on L defined by setting $\text{Min}^*(x)$ to be the meet of all the minimal primes of L exceeding x . Min^*L then denotes the subset of L consisting of all meets of minimal primes of L . We have just observed that $PL \subseteq \text{Min}^*L$. On the other hand, by [22, Lemma 2.2], each minimal prime is an upward directed supremum of polars, which implies that $\text{Min}^*L \subseteq dL$, whence $\widehat{\text{Min}^*L} \subseteq dL$. Moreover, $dL = \widehat{\text{Min}^*L}$, since each polar is an infimum of minimal primes. Finally, recall [22, Proposition 5.2], which states that dL is regular precisely when $dL = \text{Min}^*L$.

The following is a basic observation, which is probably known, but is not recorded anywhere.

Lemma 1.4. Suppose that L is an algebraic frame. Then $x \in L$ is regular if and only if every compact element $c \leq x$ is well below x .

Proof. Since L is algebraic, the sufficiency is clear. Conversely, suppose that x is regular and $c \leq x$ is compact. Then there are finitely many x_1, \dots, x_n , all well below x , such that $c \leq x_1 \vee \dots \vee x_n$. As is well known, $x_1 \vee \dots \vee x_n \leq x$, and so $c \leq x$. ■

We record here most of the characterization of regular algebraic frames from [22]; a version of this, without any mention of regularity appears as [18, Theorem 2.4].

Theorem 1.5 ([22, Theorem 2.4(a)]). Let L be an algebraic frame. The following are equivalent:

- (a) L is regular.

- (b) For each $c \in \mathfrak{k}(L)$, $c \vee c^\perp = 1$.
 (c) L has the FIP and each prime of L is minimal.

Recall as well the following usage from [22]: for any algebraic frame with the FIP L , and any inductive nucleus j on L , we say that L is j -regular if jL is regular.

For any frame L , $\text{Max}(L)$ stands for the space of maximal elements (with the relative spectral topology). Evidently, without some hypotheses on L , $\text{Max}(L)$ may very well be empty.

Definition & Remark 1.6. (a) It is well known that in a normal frame L , $\text{Max}(L)$ is a Hausdorff subspace, and that each prime of L is exceeded by at most one maximal element. If L is coherent, then the latter feature implies normality.

(b) Let L be an algebraic frame. We say that L has the *disjointification property* (or, simply, that L is a frame *with disjointification*) if for each pair of compact elements $a, b \in L$ there exist disjoint $c, d \in \mathfrak{k}(L)$ such that

1. $c \leq a$ and $d \leq b$, and
2. $a \vee b = a \vee d = c \vee b$.

It is well known that L has disjointification if and only if $\downarrow a$ is a normal frame, for each $a \in \mathfrak{k}(L)$. Hence the reason that the disjointification property is alternately referred to as *relative normality* (such as in [27]), and as *coherent normality* (in [1]). Observe that if L has disjointification and, given $a, b \in \mathfrak{k}(L)$, $c, d \in \mathfrak{k}(L)$ are chosen to witness the disjointification of a and b , then $a = (a \wedge b) \vee c$ and $b = (a \wedge b) \vee d$.

The following lemma is well known; see [24]. One should also cite [27, Lemma 2.1], where a proof is given. If $\text{Spec}(L)$ satisfies the condition of this lemma, it is called a *root system*.

Lemma 1.7. Suppose that L is an algebraic frame with disjointification. Then, for any $p \in \text{Spec}(L)$, $\uparrow p$ is a chain. The converse is true if L has the FIP.

To conclude this general introduction, we recall a construction which is rather prominent in the study of rings of continuous functions [11, 4I]. The reference for this discussion is [20].

Definition 1.8. Assume that L is an algebraic frame with the FIP. For $p \in \text{Spec}(L)$, put

$$O(p) = \vee \{a^\perp : a \in \mathfrak{k}(L), a \not\leq p\}.$$

The following is shown in [20]; we refer the reader to that article.

Proposition 1.9. Let $p, q \in \text{Spec}(L)$; then

- (a) if $p \leq q$, then $O(q) \leq O(p)$.
- (b) $O(p)$ is a d -element.
- (c) $q \in \text{Min}(L)$ and $q \leq p$ imply that $O(p) \leq q$.
- (d) If $O(p) \leq q$ and q is minimal over $O(p)$ then $q \leq p$.
- (e) $O(p) = \wedge \{q \in \text{Min}(L) : q \leq p\}$.

2. Convex ℓ -subgroups

This short section is dedicated to an exposition of the basic features of the frame $\mathcal{C}(G)$ of all convex ℓ -subgroups of a lattice-ordered group G , a frame which occurs rather prominently in the sequel. For background on lattice-ordered groups we refer the reader to [3, 7]. In this exposition all groups considered below are abelian.

Definition & Remark 2.1. For the record, $(G, +, 0, -(\cdot), \vee, \wedge)$ is a *lattice-ordered group* (abbreviated ℓ -group) if $(G, +, 0, -(\cdot))$ is a group with (G, \vee, \wedge) as an underlying lattice, and the following distributive laws holds:

$$a + (b \vee c) = (a + b) \vee (a + c).$$

The above distributive law then implies the corresponding distributive law for sum over infimum. The elements of G for which $g \geq 0$ are said to be *positive*; the set of positive elements of G is denoted G^+ .

We recite the information to be used in this article; in the sequel G stands for an ℓ -group.

1. The underlying lattice of an ℓ -group is distributive [7, Corollary 3.17], and the group structure is torsion free [7, Propositions 3.15 & 3.16].
2. A subgroup of G is called an ℓ -subgroup if it is a sublattice as well. The ℓ -subgroup C is *convex* if $a \leq g \leq b$ with $a, b \in C$ implies that $g \in C$. Let $\mathcal{C}(G)$ denote the lattice of all convex ℓ -subgroups of G . $\mathcal{C}(G)$ is a complete sublattice of the lattice of all subgroups of G [7, Theorem 7.5], and an algebraic frame; the latter is due to G. Birkhoff [7, Proposition 7.10]. $\mathcal{C}(G)$ satisfies the FIP [7, Proposition 7.15], but, in general, fails to be coherent.
In $\mathcal{C}(G)$ the convex ℓ -subgroup generated by $a \in G$ is denoted $G(a)$. Each compact element of $\mathcal{C}(G)$ is of this form; this is a restatement of [7, Proposition 7.16].
3. It is well known that, for every ℓ -group G , $\mathcal{C}(G)$ is a frame with disjointification. Indeed, if $a, b \geq 0$ in G , let $c = a - (a \wedge b)$ and $d = b - (a \wedge b)$; then $G(c)$ and $G(d)$ witness the disjointification of $G(a)$ and $G(b)$. Recall that, since $\mathcal{C}(G)$ is an algebraic frame with the FIP and disjointification, $\text{Spec}(\mathcal{C}(G))$ is a root system.
4. G is said to be *hyperarchimedean* when $\mathcal{C}(G)$ is a regular frame. The topic of hyperarchimedean ℓ -groups was first developed by Conrad in [6].

3. Regularity revisited

Throughout this section it is assumed that L is an algebraic frame with the FIP.

In [22, 2.1] it is noted that regular elements are always d -elements. This, in part, motivated the introduction there of several conditions concerning regularity, listed from strongest to weakest:

- Reg(1) L is regular.
- Reg(2) Each d -element of L is regular.
- Reg(3) Each polar of L is regular.
- Reg(4) Each c^\perp , with c compact, is regular.

We have already highlighted, in Theorem 1.5, a number of conditions which are equivalent to Reg(1). Parts (b) and (c) of [22, Theorem 2.4] characterize Reg(2) and Reg(3), respectively, and we summarize that presently. The reader who is familiar with the literature on ℓ -groups, will recognize condition (ii) in Theorem 3.1(a) below. As used for ℓ -groups, we say here that L is *projectable* if Theorem 3.1(a)(ii) holds.

Theorem 3.1. (a) Reg(2) and Reg(3) are equivalent, and also equivalent to each of the following:

- (i) For each compact c in L , $c^{\perp\perp}$ is regular.
 - (ii) For each compact c in L , $c^{\perp\perp}$ is complemented.
- (b) Reg(4) is equivalent to each of the following:
- (i) For each pair of disjoint compact elements a and b , $1 = a^\perp \vee b^\perp$.
 - (ii) For any two distinct minimal primes p and q , $p \vee q = 1$.

The following definition is suggested by the terminology of [1].

Definition 3.2. Define $x \in L$ to be *zero-dimensional* if it is the supremum of complemented compact elements. Let $\mathfrak{z}(L)$ denote the subset of all zero-dimensional elements of L .

The reader may easily verify items (a) and (b) in the following proposition. We sketch the proof of (c). It is made smoother by first recalling the following.

A distributive lattice D with least element 0 is *relatively complemented* if for each $a \leq b \in D$, there is a $c \in D$ such that $c \wedge a = 0$ and $c \vee a = b$.

- Proposition 3.3.** (a) $x \in L$ is zero-dimensional if and only if for each compact $a \leq x$ there is a complemented compact $e \leq x$ such that $a \leq e$.
- (b) $\mathfrak{z}(L)$ is closed under formation of arbitrary suprema and finite infima. Indeed, $\mathfrak{z}(L)$ is the subframe generated by all the complemented compact elements of L .
- (c) $\mathfrak{z}(L)$ is algebraic and regular, whence $\mathfrak{z}(L) \subseteq \varrho(L)$.

Proof. (c) Note first that, since the compactness of an element is inherited by passing to a subframe, it is clear that $\mathfrak{z}(L)$ is algebraic, and that $\mathfrak{k}(\mathfrak{z}(L))$ consists precisely of the compact complemented elements of L .

By [22, Proposition 2.8], it suffices to show that $\mathfrak{k}(\mathfrak{z}(L))$ is relatively complemented; checking the latter is an easy exercise. ■

The foregoing sets up the following result. We shall refer to a frame homomorphism which takes compact elements to compact elements as a *coherent map*.

Proposition 3.4. *The inclusion $\mathfrak{z}(L) \subseteq L$ defines a coreflection from the category $\mathfrak{A}\mathfrak{F}\mathfrak{rm}$ of all algebraic frames with the FIP, together with all coherent maps, into the full subcategory $\mathfrak{Z}\mathfrak{F}\mathfrak{rm}$ of all regular algebraic frames.*

Proof. There are two elements to the proof. First, the object-values of this functor are in $\mathfrak{Z}\mathfrak{F}\mathfrak{rm}$; this follows from [Theorem 1.5](#), as the regular algebraic frames are precisely the algebraic frames whose compact elements are complemented.

The other aspect of the proof is that the inclusion $\mathfrak{z}(L) \subseteq L$ does define a coreflection. First, \mathfrak{z} is functorial, as the image of any compact and complemented element is, likewise, compact and complemented. Second, if $g : R \rightarrow L$ is any coherent frame map out of a regular algebraic frame, then for any $a \in \mathfrak{k}(R)$, $g(a)$ is compact and complemented (invoking [Theorem 1.5](#) once more), and therefore $g(a) \in \mathfrak{z}(L)$, which suffices to conclude that g factors through $\mathfrak{z}(L)$, and, thus, that $\mathfrak{z}(L) \subseteq L$ defines a coreflection, as promised. ■

In commutative algebra and homological algebra there is the notion of a pure subobject, usually defined in connection with tensor products. In our circumstances purity generalizes zero-dimensionality, and, indeed, turns out to be equivalent to regularity, as is shown in the lemma below. We refer the reader especially to De Marco [9] for a discussion of purity in the context of ordered algebraic structures.

Lemma 3.5. *$x \in L$ is regular if and only if*

$$x \leq p \implies x \leq O(p).$$

*In particular, x is regular if and only if it is an infimum of $O(p)$'s; hence $\text{Reg}(L) \subseteq \text{Min}^*L$.*

Proof. First assume that $x \in L$ is regular, and that $x \leq p \in \text{Spec}(L)$. If $x \not\leq O(p)$, there is a compact $a \leq x$ – since x is regular – such that $a \not\leq O(p)$. But then, on the one hand, $a^\perp \vee x = 1$, while $a^\perp \leq p$, which evidently leads to a contradiction. We conclude that $x \leq O(p)$.

Conversely, suppose x satisfies the condition in the lemma. Let $c \leq x$, with $c \in \mathfrak{k}(L)$. If $c^\perp \vee x < 1$, there is a prime p which exceeds $c^\perp \vee x$. This implies that $c \not\leq O(p)$. On the other hand, $x \leq p$, whence $x \leq O(p)$, which is absurd. Thus, $c^\perp \vee x = 1$, proving that x is regular. The final claim is then obvious; see [1.3\(b\)](#). ■

Recall (for an arbitrary frame L) that $\text{FP}(L)$ (resp. $\text{CP}(L)$) stands for the subframe (resp. complete sublattice) of L generated by PL . In general these two constructs are different [[20](#), Example 3.2], and $dL \subseteq \text{FP}(L) \subseteq \text{CP}(L)$, but if $dL = \text{FP}(L)$ then also $\text{FP}(L) = \text{CP}(L)$ [[20](#), 1.3].

[Lemma 3.5](#) and the foregoing remarks about $\mathfrak{z}(L)$ have the following consequence. We postpone a more detailed analysis of the inclusions in [\(†\)](#) below until Section 8.

Proposition 3.6. *The following inclusions hold for any algebraic frame L with the FIP:*

$$\mathfrak{z}(L) \subseteq \varrho(L) \subseteq \text{Reg}(L) \subseteq \text{Min}^*L \subseteq dL \subseteq \text{FP}(L) \subseteq \text{CP}(L). \quad (\dagger)$$

4. When “well below” interpolates

Throughout this section L stands for an algebraic frame with the FIP. We return to a consideration of the subframe $\text{Reg}(L)$ of the frame L ; the reader is referred to the discussion in [1.1.6](#) and [1.1.7](#). In particular, if L is normal then \leq interpolates, and then it can be shown – and we shall do so presently, for completeness – that then $\text{Reg}(L)$ is regular, and, thus, $\varrho(L) = \text{Reg}(L)$.

We want more detail, however: the inclusion of $\text{Reg}(L)$ in L has a dual map, whose restriction to $\text{Spec}(L)$ has an interesting description (subject to some hypotheses).

Remark 4.1. Let L be a frame. It has been observed, $\text{Reg}(L)$ is a subframe of L . The assignment $i_* : L \rightarrow \text{Reg}(L)$ defined by

$$i_*(x) = \vee \{y \in \text{Reg}(L) : y \leq x\},$$

is the right adjoint of the inclusion $i : \text{Reg}(L) \longrightarrow L$. In taking this point of view one regards the two frames as categories; there is an arrow $x \rightarrow y$ precisely when $x \leq y$ [12, 27Q]. In this setting infima are limits and suprema are colimits, and thus – by [12, Theorem 28.11] – i_* preserves arbitrary infima. As a right adjoint, i_* also induces a map $\text{Spec}(i) : \text{Spec}(L) \longrightarrow \text{Spec}(\text{Reg}(L))$, by restriction. $\text{Spec}(i)$ is continuous relative to the hull–kernel topologies on the two spaces.

The final preliminary to Theorems 6.3 and 7.1 is the auxiliary map $x \mapsto \bar{x}$ defined by

$$\bar{x} = \vee \{c \leq x : c \in \mathfrak{k}(L)\},$$

and the lemma which follows.

Lemma 4.2. *Suppose that L is an algebraic frame with the FIP. Then for each $x \in L$, $i_*(x) = \bar{x}$.*

Proof. Lemma 1.4 implies that $i_*(x) \leq \bar{x}$. For the converse, imitate the proof of Lemma 1.4 to show that if d is compact and below \bar{x} , then it is well below \bar{x} ; then apply that lemma again to conclude \bar{x} is regular, and the desired result follows. ■

It seems to be folklore that, when “well below” interpolates, $\text{Reg}(L)$ is regular, and, hence, equal to $\varrho(L)$. Here this fact is an easy consequence of the foregoing.

Proposition 4.3. *Suppose that \leq interpolates in L . Then $\text{Reg}(L)$ is regular.*

Proof. Suppose that x is regular and $c \in \mathfrak{k}(L)$ satisfies $c \leq x$. Then $c \leq x$, and, by our assumption, there is a $y \in L$ such that $c \leq y \leq x$. Then too, $c \leq i_*(y) \leq y \leq x$, and it follows that x is the supremum over all $i_*(y)$, for all interpolating $y \leq x$. ■

We conclude this section with an observation regarding the sequence (†) in Proposition 3.6.

Proposition 4.4. *Suppose L is projectable. Then $\mathfrak{z}(L) = \varrho(L) = \text{Reg}(L)$.*

Proof. If $x \in L$ is regular, then by Lemma 1.4,

$$x = \vee \{c : c \leq x\} = \vee \{c^{\perp\perp} : c \leq x\},$$

and each $c^{\perp\perp}$ is complemented and compact in dL ; that is, $x \in \mathfrak{z}(dL)$. This shows that $\text{Reg}(L) \subseteq \mathfrak{z}(dL)$.

On the other hand, since L is projectable, $(a \vee b)^{\perp\perp} = a^{\perp\perp} \vee b^{\perp\perp}$, for any pair of compact elements a, b . Thus, dL is a subframe of L , and using the monoreflective property of \mathfrak{z} , we have that,

$$\text{Reg}(L) \subseteq \mathfrak{z}(dL) \subseteq \mathfrak{z}(L) \subseteq \text{Reg}(L),$$

which completes the proof. ■

5. Completely distributive frames

We describe the regular elements of a completely distributive frame. It is assumed in this section that L is algebraic with the FIP. To say that L is *completely distributive* means that for each pair of index sets I and J , and elements $x_{i,j} \in L$, we have

$$\bigwedge_{i \in I} \bigvee_{j \in J} x_{i,j} = \bigvee_{f \in J^I} \bigwedge_{i \in I} x_{i,f(i)}.$$

We recall the so-called “Conrad Program” for completely distributive frames with disjointification. We refer the reader to [27], and to [17] for a slightly more general account. The work in these references harkens back to the inspiration provided by [5]. The following theorem collects some of the principal elements of that program.

Theorem 5.1. *Suppose L has disjointification. Then the following are equivalent.*

(a) L is completely distributive.

(b) L is a dual frame; that is, for each $S \subseteq L$

$$a \vee \left(\bigwedge S \right) = \bigwedge \{a \vee s : s \in S\}.$$

(c) For each $c \in \mathfrak{k}(L)$, $\downarrow c$ has a finite number of maximal elements.

To give a thorough description of the regular elements here, we must first establish some terminology associated with $\text{Spec}(L)$. We assume L has disjointification, and, hence, that $\text{Spec}(L)$ is a root system.

Definition 5.2. (a) The primes p and q of L are said to be *linked* if there is an $r \in \text{Spec}(L)$ which is an upper bound. The “linkage” relation is clearly reflexive and symmetric, and since $\text{Spec}(L)$ is a root system, it is also transitive.

An equivalence class of the linkage relation is called a *component* of $\text{Spec}(L)$, and $\text{Spec}(L)$ is said to be *connected* if any two primes are linked.

(b) If $v \in L$ is maximal with respect to not exceeding some $c \in \mathfrak{k}(L)$, then we say that v is a *value* of c . It is well known that values are prime, and, by Zorn’s Lemma, every nonzero compact element has a value. The compact elements described in (c) of Theorem 5.1 are precisely the elements having finitely many values. If c has exactly one value v , then we say c and v are *special*, and also that c is *special at* v .

It is well known [27] that the conditions in Theorem 5.1 are also equivalent to each of these:

1. every value of L is special;
2. each compact element of L has finitely many values;
3. each compact element of L is a disjoint supremum of special compact elements.

Here is the characterization of $\text{Reg}(L)$ in this context. Let SL denote the subset of complemented elements of L . It is well known that SL is a boolean algebra, and, indeed, a subalgebra of PL .

Theorem 5.3. Suppose L is completely distributive, with disjointification. Then $\text{Reg}(L) = SL$. Moreover:

1. $\text{Reg}(L)$ is regular, and a complete atomic boolean algebra, with $\text{Spec}(\text{Reg}(L))$ discrete.
2. The atoms of $\text{Reg}(L)$ are described as follows: Let S denote the set of components of $\text{Spec}(L)$. For each $S \in S$,

$$\begin{aligned} x_S &= \bigwedge \{O(p) : p \in T, T \neq S, T \in S\} \\ &= \bigvee \{c \in \mathfrak{k}(L) : c \text{ special at } v \in S\}. \end{aligned}$$

Proof. Clearly, every complemented element is regular. Conversely, if $x \in \text{Reg}(L)$ write

$$x = \vee \{c \in \mathfrak{k}(L) : c \leq x\}.$$

Then, as L is a dual frame,

$$x \vee x^\perp = x \vee \left(\bigwedge_{c \leq x} c^\perp \right) = \bigwedge_{c \leq x} x \vee c^\perp = 1.$$

Since $\text{Reg}(L) = SL$, it is clear that $\text{Reg}(L)$ is regular, and it is well known that a completely distributive boolean frame must be atomic. For the rest, let us put

$$x_S \equiv \bigvee \{c \in \mathfrak{k}(L) : c \text{ is special at some } v \in S\},$$

and also the companion

$$y_S \equiv \bigvee \{c \in \mathfrak{k}(L) : c \text{ is special at some } v \notin S\}.$$

Now, it is easy to see that $x_S \wedge y_S = 0$, since any two special elements having incomparable values are disjoint; note that the latter uses the fact that $\text{Spec}(L)$ is a root system. Further, in view of the comments in 5.2(b), each $a \in \mathfrak{k}(L)$ may be written as $a = a_S \vee b_S$, where a_S is the supremum of the special factors of a having their values on the component S , and b_S is the supremum of the remaining special factors of a (whose values lie off S). This suffices to show that $x_S \vee y_S = 1$, and, thus, that x_S is complemented.

To see that x_S is an atom of $\text{Reg}(L)$, observe that if c and d are any two special elements $\leq x_S$, then their respective values, v and w , lie on S , so that they have a common upper bound $p \in S$, and, consequently, there is a special element $b \leq x_S$ which bounds c and d above.

Further, and by a similar argument, if $x \in \text{Reg}(L)$ and c and d are special elements such that $c \leq x$ and $d \leq x^\perp$, then the values of c and d must lie on distinct components of $\text{Spec}(L)$. The reader will readily verify that x is then a join of elements of the form x_S , and, in particular, that each atom is of that form.

Finally, we leave the verification that

$$x_S = \bigwedge \{O(p) : p \in T, T \neq S, T \in \mathcal{S}\}$$

to the reader. Then the proof of this theorem is complete. ■

6. Coherent normal frames

A great deal is known about coherent normal frames. The reader is referred to [1,2,23]. Below we will specify the connections to [1] and [2]. It is assumed for the rest of this section that L is a coherent frame. Note that, by Zorn's Lemma, every $x < 1$ lies beneath an $m \in \text{Max}(L)$; furthermore, $\text{Max}(L)$ is a compact space.

Proposition 6.1. *Suppose that L is normal, and $m, n \in \text{Max}(L)$.*

- (a) *If m and n are distinct, then $O(m) \vee O(n) = 1$.*
- (b) *$O(m)$ is regular.*
- (c) *If x is regular then it is the infimum of the $O(m)$, with $m \in \text{Max}(L)$ and $x \leq m$.*

Proof. (a) As a preliminary step, we establish that $m \vee O(n) = 1$. Using the assumption of normality, we have $a \leq m$ and $b \leq n$ with $a \wedge b = 0$ and $1 = a \vee n = m \vee b$. Clearly, $a \not\leq n$, which means that $b \leq O(n)$, and it follows that $m \vee O(n) = 1$.

Now if $O(m) \vee O(n) < 1$, there is a maximal $q \geq O(m) \vee O(n)$. By the preceding paragraph, $q = m$, and, similarly, $q = n$, a contradiction.

(b) If $p \in \text{Spec}(L)$ and $O(m) \leq p$, then $p \leq n$, for a suitable maximal n . By (a), $n = m$, and, thus, $O(m) \leq O(p)$, which proves that $O(m)$ is regular.

(c) is routine and is left to the reader. ■

Under the assumptions of Proposition 6.1, $\text{Spec}(i)$ is none but $p \mapsto O(m)$, where m is the unique maximal element of L exceeding p . That is the upshot of the next proposition; the proof is omitted.

Proposition 6.2. *Suppose L is normal. For each $m \in \text{Max}(L)$, $O(m) \in \text{Max}(\text{Reg}(L))$. Further, $\text{Spec}(i)(p) = O(m)$, for each $p \in \text{Spec}(L)$, where m is the unique maximal element exceeding p .*

The main result of this section gives a fairly complete account of the map $\text{Spec}(i)$. It is the kind of description one should like to generalize; unfortunately this is not easy. There are some partial results which, however, are best left for a future discussion.

We have already noted that the final claim in Theorem 6.3 is known for any normal frame.

Theorem 6.3. *Suppose that L is normal. Then*

- (a) *for each $y \in \text{Reg}(L)$ there is an $z \in L$ maximal with respect to the property $y = i_*(z)$;*
- (b) *$\text{Spec}(i)$ maps onto $\text{Spec}(\text{Reg}(L))$, and the restriction to $\text{Max}(L)$ is a homeomorphism.*

It follows that $\text{Reg}(L)$ is regular, and $\varrho(L) = \text{Reg}(L)$.

Proof. We begin the proof of (a) by applying Lemma 4.2: for each $x \in L$,

$$i_*(x) = \vee \{c \in \mathfrak{k}(L) : c \preceq x\}.$$

An application of Zorn's Lemma to the fiber $i_*^{-1}(x)$, proves (a); we shall go through the argument, as there is a point to be made at the conclusion of the proof (Remark 6.4).

We suppose that $y \in \text{Reg}(L)$; since $y \in i_*^{-1}(\{y\})$, this fiber is nonempty. Now suppose that $\{x_\lambda : \lambda \in \Lambda\}$ is a chain in this fiber, and set $x = \vee_\lambda x_\lambda$. We now show that $i_*(x) = y$: for suppose that $c \in \mathfrak{k}(L)$ and $c \preceq x$; then $c^\perp \vee x = 1$, and since 1 is compact, we have that $c^\perp \vee x_\mu = 1$, for suitable $\mu \in \Lambda$. This implies that $c \preceq x_\mu$, and the reader will easily see (by Lemma 4.2) that $i_*(x) = \vee_{\lambda \in \Lambda} i_*(x_\lambda) = y$, as promised.

Then, by Zorn's Lemma we may choose z maximal with respect to $i_*(z) = y$, and (a) is proved.

As to (b), if q is prime in $\text{Reg}(L)$, choose $p \in L$ maximal with respect to $i_*(p) = q$; we show that p is prime in L . Recall that $i_*(a \wedge b) = i_*(a) \wedge i_*(b)$; thus, if $a \wedge b = p$, then $i_*(a) \wedge i_*(b) = q$, whence either $i_*(a) = q$ or $i_*(b) = q$. By the maximality of p relative to $i_*(p) = q$, it follows that either $a = p$ or $b = p$, proving that $p \in \text{Spec}(L)$. This proves $\text{Spec}(i)$ is surjective.

From Proposition 6.2 it is clear that $\text{Spec}(\text{Reg}(L))$ is an antichain, and hence a compact T_1 space. It is then an easy exercise to show that $\text{Spec}(\text{Reg}(L))$ is compact Hausdorff. Since the restriction of $\text{Spec}(i)$ to $\text{Max}(L)$ is one-to-one (Proposition 6.1), it is also a homeomorphism.

Finally, as $\text{Reg}(L)$ is spatial, this suffices to show that $\text{Reg}(L)$ is regular. ■

Remark 6.4. The proof of (b) in Theorem 6.3 shows that i_* preserves joins of chains; it is easily seen that it also preserves up-directed suprema. However, the coherence of the frame appears to be important. Example 7.2 shows that, in general, this property fails.

Banaschewski, in [1,2], approaches the above presentation of $\varrho(L) = \text{Reg}(L)$, for compact normal frames, constructively, from the point of view of the saturation nucleus. We comment briefly, next.

Remark 6.5. Let L be a compact frame. The *saturation nucleus* s is defined by the rule

$$s(x) = \vee \{a \in L : a \vee y = 1 \Rightarrow x \vee y = 1\},$$

which is the largest element of L such that $s(x) \vee y = 1$ implies that $x \vee y = 1$, by a routine application of the compactness of 1. The map s does define a nucleus, and (with the assumption of Zorn's Lemma), it is none other than Max^* , given by

$$\text{Max}^*(x) = \bigwedge \{m \in \text{Max}(L) : m \geq x\}.$$

(The reader is referred to [1, Section 1], and, for further reading on Max^* , to [23].)

It is shown in [1] that if L is also normal, then sL is isomorphic to the subframe $\varrho(L)$. Moreover, s has a right inverse r , which is the restriction to sL of i_* . Indeed, the situation is adequately captured by the following commutative square:

$$\begin{array}{ccc} L & \xrightarrow{s} & sL \\ i_* \downarrow & \searrow \cong & \downarrow r \\ \varrho(L) & \xrightarrow{i} & L \end{array}$$

Note that the diagonal isomorphism is the map r itself, with the codomain restricted to $\varrho(L)$.

Finally, the material in [2, Section 2] goes on to show that, for compact normal frames, the spectrum of sL is canonically isomorphic to $\text{Max}(L)$, and that the saturation nucleus is, in fact, functorial.

There is a general principle, first articulated in [20], which is useful in deciding whether $\text{Reg}(L)$ is regular. We recall it next, as it applies to the present context.

Let j be an inductive nucleus on L (1.1.9). Recall [19] that a compact element a is said to be a j -unit if $j(a) = 1$. We then have the following straightforward result.

Proposition 6.6. Suppose that j is an inductive nucleus such that every polar is a j -element, and jL is a sublattice of L . Then jL is a complete sublattice, and $\text{Reg}(L) = \text{Reg}(jL)$. If there is a j -unit then $\text{Reg}(L)$ may be computed in a coherent frame.

Proof. First, jL is a complete sublattice because it is inductive; i.e., closed under up-directed suprema. Next, the assumption that jL contains PL implies that

$$x^\perp = j(x^\perp) = j(x)^\perp.$$

Thus, if $x \leq y$ then $j(x) \leq y$, which is enough to show that $\text{Reg}(L) \subseteq jL$, and, therefore, that $\text{Reg}(L) = \text{Reg}(jL)$. ■

Remark 6.7. All rings considered in this discussion are commutative, with identity.

Suppose that A is an f -ring; that is, A is an ℓ -group which is at once a ring, such that for each disjoint pair $f \wedge g = 0$ and each $h \geq 0$, we have $fh \wedge g = 0$. Consider the nucleus ℓ , which computes, given $C \in \mathcal{C}(A)$ the ℓ -ideal $\ell(C)$ generated by C . (We explain the term ℓ -ideal : I is both a ring ideal and a convex ℓ -subgroup.) It is well known and easy to check that

$$\ell(C) = \{r \in A : |r| \leq fa, \text{ for some } 0 \leq f \in A, 0 \leq a \in A\}.$$

ℓ is an inductive nucleus and the frame $\mathcal{C}_\ell(A) \equiv \mathcal{C}_\ell(A)$ of ℓ -ideals is a sublattice of $\mathcal{C}(A)$, as the sum of ℓ -ideals is always an ℓ -ideal. Moreover, every polar is an ℓ -ideal, by definition of “ f -ring”.

The significance of Proposition 6.6 here is that, if we are interested in the regular elements of a frame of convex ℓ -subgroups of an f -ring A , then we may as well assume $L = \mathcal{C}_\ell(A)$. Note that the identity 1 of A generates A as an ℓ -ideal; that is, $\mathcal{C}_\ell(A)$ is coherent.

Since $\mathcal{C}(A)$ has disjointification, $\mathcal{C}_\ell(A)$ is normal. Therefore, the consequence of these observations, along with Theorem 6.3 is this.

Corollary 6.8. Suppose that A is an f -ring. Then $\text{Reg}(\mathcal{C}(A))$ is regular.

For another application of the above shifting technique, we must first recover some information about the frame of d -elements in $\mathcal{C}(G)$. Recall, in any algebraic frame L , that $x \in L$ is a d -element if $c \leq x$, with $c \in \mathfrak{k}(L)$, implies that $c^{\perp\perp} \leq x$ (1.3(b)). Put $\mathcal{C}_d(G) \equiv d\mathcal{C}(G)$, for each ℓ -group G . The members of $\mathcal{C}_d(G)$ are called d -subgroups.

Remark 6.9. For purposes of this discussion we shall assume that G stands for a vector lattice; that is an abelian ℓ -group which is simultaneously a real vector space, such that, for each positive $r \in \mathbb{R}$ and $g \in G^+$, we have $rg \in G^+$. We also assume that G is archimedean. We refer the reader to [25], as well as to [8] for the more specific context of rings of continuous functions.

Recall that the sequence $(g_n)_{n < \omega}$ in G o -converges to $g \in G$ if there is a decreasing sequence u_n with $\bigwedge_n u_n = 0$ such that $|g - g_n| \leq u_n$. The sequence $(g_n)_{n < \omega}$ is o -Cauchy if there is a decreasing sequence u_n with $\bigwedge_n u_n = 0$ such that, for every $k \in \mathbb{N}$, $|g_{n+k} - g_n| \leq u_n$. Finally, G is order complete if every o -Cauchy sequence of G o -converges.

In [25, Theorem 11.2] it is shown that the uniformly complete vector lattice G is order complete precisely when the sum of two d -subgroups is a d -subgroup, and, hence, if and only if $\mathcal{C}_d(G)$ is a complete sublattice of $\mathcal{C}(G)$. It is also well known that any order complete vector lattice is uniformly complete. For a reminder on the subject of uniform completeness in ℓ -groups we refer the reader to [13, Section 2].

Using Theorem 6.3 and Proposition 6.6 again, as before, one gets the following corollary. Recall that, in an ℓ -group G , $u \in G$ is a weak order unit provided $|u| \wedge g = 0$ implies that $g = 0$. Note that G has a weak order unit if and only if $\mathcal{C}_d(G)$ is coherent.

Corollary 6.10. Suppose that G is an order complete vector lattice with a weak order unit. Then $\text{Reg}(\mathcal{C}(G))$ is regular.

7. Fibers of $\text{Spec}(i)$

Even with the assumption of disjointification, the regularity of $\text{Reg}(L)$ is not easy to resolve; there are several minor generalizations of the results in Section 6, but a presentation of them here seems premature. Rather, this section is devoted to a result, which, although technical, can be formulated rather straightforwardly. It also represents our best efforts to get to the inner workings of $\text{Spec}(i)$. More to the point, it yields an example of an algebraic frame L with the FIP and disjointification such that $\text{Reg}(L)$ is not regular.

The reader is reminded that L is d -regular if the frame dL is regular. Further, recall that when L has disjointification, $\text{Spec}(L)$ is a root system; that is, no two incomparable primes have a common lower bound.

Theorem 7.1. Suppose that L is an algebraic frame with the FIP and disjointification. If L is also d -regular, then $\text{Spec}(i)(q) \geq \text{Spec}(i)(p)$ if and only if q lies in the closure of the component C of p . In particular, primes lying in the same component have the same image under $\text{Spec}(i)$.

Proof. Since the last assertion evidently follows from the first, one should only have to prove the first claim. Yet the validity of the final claim does not depend on d -regularity, whereas the stronger one appears to. For this reason we explicitly establish the final claim first. Note that implicit use is made of [Lemma 4.2](#), throughout; that is,

$$\text{Spec}(i)(p) = \vee \{c \leq p : c \in \mathfrak{k}(L)\},$$

for each prime p of L .

(a) First, if the primes p and q are linked then $\text{Spec}(i)(q) = \text{Spec}(i)(p)$.

For each compact $c \leq q$, $c^\perp \vee q = 1$. If this identity fails for p , then there is a prime $m \geq c^\perp \vee p$, which means that m is linked to p and therefore to q . By disjointification, m and q have a common upper bound n . Since $c^\perp \leq n$, this contradicts $c^\perp \vee q = 1$. Thus, $c^\perp \vee p = 1$, and hence $c \leq p$. By symmetry the converse holds, which implies that $\text{Spec}(i)(q) = \text{Spec}(i)(p)$.

(b) Continuity of $\text{Spec}(i)$ implies that if $q \in \text{cl } C$, then $\text{Spec}(i)(q) \geq \text{Spec}(i)(p)$.

In fact, if $q \in \text{cl } C$, then, by elementary properties of the hull–kernel topology, we have $q \geq \bigwedge C$, and this implies the desired result.

(c) Suppose that $q \notin \text{cl } C$: that is, there is a compact $a \in L$ such that $a \not\leq q$, but for each prime m linked to p , $a \leq m$. Claim: $a^\perp \vee m = 1$, for each such m .

Suppose not; then there is a prime $n \geq a^\perp \vee m$, and, evidently, n is also linked to p . Now, there is a prime $n_0 \leq n$ – and therefore linked to p as well – which is minimal with respect to $a^\perp \leq n_0$, and – this is the sole use of d -regularity! – by [\[22, Proposition 5.2\]](#), it follows that $n_0 \in \text{Min}(L)$. This is absurd, as $a, a^\perp \leq n_0$.

(d) Thus, if $q \notin \text{cl } C$, then the compact a in (c) satisfies $a \leq p$, while $a \not\leq q$, which implies that $\text{Spec}(i)(q) \not\leq \text{Spec}(i)(p)$.

This concludes the proof of the theorem. ■

Finally, we illustrate with two examples. The first shows that $\text{Reg}(L)$ need not be regular, even with disjointification, whereas the second modifies the first slightly, to show that one cannot simply drop the hypothesis of d -regularity in [Theorem 7.1](#). Both examples rely on a fundamental construction in lattice-ordered groups. In [\[21\]](#) it is the principal tool in the production of algebraic frames with the FIP and disjointification, which fail to be normal.

Example 7.2. We refer the reader to [Section 2](#). The example L in question here is the frame of all convex ℓ -subgroups $\mathcal{C}(G)$ of an ℓ -subgroup G of the Hahn group of real-valued functions $V = V(\Lambda, \mathbb{R})$ over the root system Λ described below.

To make the reader's job easier, let us first recall what a Hahn group is. Now, $V(\Lambda, \mathbb{R})$ is the lattice-ordered group of all real-valued functions f on Λ , for which

$$\text{coz}(f) = \{\lambda \in \Lambda : f(\lambda) \neq 0\}$$

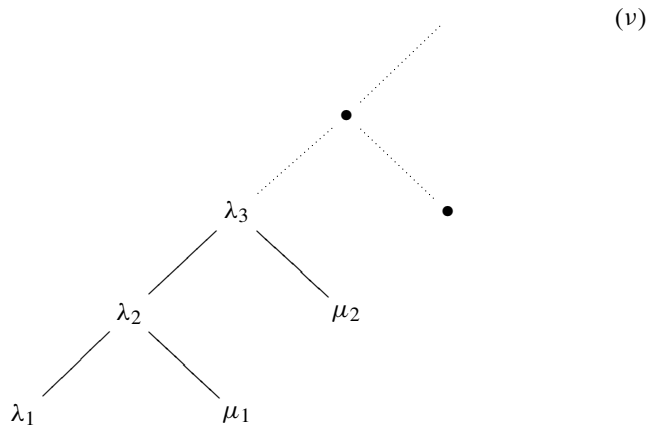
satisfies the ascending chain condition. The lattice-ordering is defined by: $f > 0$ if $f(\mu) > 0$ for each maximal element $\mu \in \text{coz}(f)$.

One associates a point $\lambda \in \Lambda$ with a convex ℓ -subgroup of $V = V(\Lambda, \mathbb{R})$ via the association

$$\lambda \mapsto \{f \in V : f(\delta) = 0, \forall \delta \geq \lambda\}.$$

The indicated subgroup is prime because Λ is a root system.

The reader is referred to [7, Theorem 51.3] for additional information on the subject.



$G \subseteq V(A, \mathbb{R})$ is defined to be the group consisting of all functions f such that $f(\lambda_n)$ is eventually 0 and $f(\mu_n)$ is eventually constant. (Note that a sequence $(r_n)_{n < \omega}$ of real numbers is *eventually* r provided there is a natural number k , such that, for all $n \geq k$, we have that $r_n = r$; such sequences are also said to be *eventually constant*.)

There are two components to $\text{Spec}(\mathcal{C}(G))$: Λ itself and the single prime ν in the upper right-hand corner of the diagram (in the parentheses). The latter corresponds to the subgroup of G of all functions f which are eventually zero on both sequences, (λ_n) and (μ_n) . This prime lies in the closure of Λ . Notice that ν is also both minimal and maximal; thus $O(\nu) = \nu$, and so ν is regular.

On the other hand, as $\nu \in \text{Max}(\text{Spec}(\mathcal{C}(G)))$, we get that $\{\nu\}$ is closed. Finally, it is easy to show that $\mathcal{C}(G)$ is d -regular. Applying Theorem 7.1, we have (in $\text{Reg}(\mathcal{C}(G))$) that

$$1 > \nu = \text{Spec}(i)(\nu) > \text{Spec}(i)(\Lambda) = \{0\},$$

the latter equality being a consequence of the fact that, for each $\lambda \in \Lambda$, we have $\text{Spec}(i)(\lambda) \leq \bigwedge \Lambda = 0$ (Lemma 3.5).

We summarize:

1. $\text{Reg}(\mathcal{C}(G)) = \{0, \nu, 1\}$, which is not regular.
2. It is shown in [21] that L is not normal. This follows from the foregoing, as in normal frames $\text{Reg}(L)$ is regular.
3. In L , $1 = \bigvee_n \lambda_n$, yet while $i_*(1) = 1$, each $i_*(\lambda_n) = 0$, showing (as promised in 6.4) that i_* can fail to preserve joins of chains.

Example 7.3. Let $H = G \times \mathbb{R}$, where G is the ℓ -group in Example 7.2. The group operation in H is coordinatewise, and in H we put $(g, r) \geq 0$ if $g \geq 0$, with $g(\mu_n)$ eventually positive, and if $g(\mu_n)$ is eventually 0, then $r \geq 0$. This time $L = \mathcal{C}(H)$; note that the addition of the new factor has the effect of placing a prime element ν_0 under the prime labeled ν in the above diagram.

We leave it to the reader to check that

1. L is not d -regular: the convex ℓ -subgroup generated by the element $(0, 1)$ in H does not have a relative complement in the convex ℓ -subgroup generated by $(u, 0)$, where $u(\lambda_n) \equiv 0$ and $u(\mu_n) \equiv 1$.
2. $\text{Spec}(L)$ has two components, Λ and the chain $\nu > \nu_0$; however, ν lies in the closure of Λ , whereas ν_0 does not. Still, $\text{Spec}(i)(\nu_0) = \nu_0 > 0 = \text{Spec}(i)(\lambda)$, for each $\lambda \in \Lambda$.
3. $\text{Reg}(L) = \{0, \nu_0, 1\}$, which is verified as in the previous example.

8. Distinctions

Here we analyze the inclusions in Proposition 3.6 (†). They are all proper, in general, and we subject the coincidences to closer examination as well. Unless the contrary is specified, L stands throughout this section for an algebraic frame with the FIP. We begin by noting two examples.

Example 8.1. A polar which is not regular.

Let $L = \mathcal{C}(G)$, where $G = C(\alpha\mathbb{N})$, and $\alpha\mathbb{N}$ denotes the one-point compactification of the discrete set of natural numbers. Observe that G is the group of all convergent real sequences.

Let P be the set of functions in G which are zero at the even coordinates. Let M denote the set of sequences which converge to zero. Then $P \subseteq M$, but $P \not\subseteq O(M)$, whence P is not regular.

Example 8.2. A regular element which is not zero-dimensional.

$L = \mathcal{C}(G)$, where $G = C([0, 1])$, and $[0, 1]$ denotes the closed unit interval with the usual topology. Let $M = M_1$, the ideal of functions which vanish at 1; $O(M)$, consisting of the functions which vanish on a neighborhood of 1, is regular, but, since the underlying space is connected, it is clearly not zero-dimensional.

Recall that $\text{FP}(L)$ (resp. $\text{CP}(L)$) stands for the subframe (resp. complete sublattice) of L generated by PL .

Remark 8.3. We take up the inclusions of Proposition 3.6 (†) from left to right, excluding consideration of the inclusions $dL \subseteq \text{FP}(L) \subseteq \text{CP}(L)$, as they are dealt with in [20].

(a) $\mathfrak{z}(L) \subseteq \varrho(L)$: This is proper; see Example 8.2. In the example in question the frame is coherent and has disjointification, which, according to Theorem 6.3 implies that $\varrho(L)$ and $\text{Reg}(L)$ coincide.

(b) $\varrho(L) \subseteq \text{Reg}(L)$: Equality fails, in general, even with disjointification; we refer the reader to Example 7.2. We summarize what we do know; $\text{Reg}(L)$ is regular provided one of the conditions below are satisfied:

1. L is completely distributive and has disjointification (Theorem 5.3).
2. L is projectable (Proposition 4.4).
3. L is normal (as \leq interpolates in this case).

(c) $\text{Reg}(L) \subseteq \text{Min}^*L$: This is also proper; see Example 8.1. In Proposition 8.4, below, it is shown that if equality holds here, then $\text{Reg}(L) = \text{Min}^*L = \text{FP}(L) = \text{CP}(L)$.

(d) $\text{Min}^*L \subseteq dL$: Equality holds if and only if dL is regular; see 1.3(b).

Proposition 8.4. The following are equivalent for L :

- (a) $\text{Reg}(L) = \text{Min}^*L$.
- (b) Every polar is regular, i.e., $\text{Reg}(2)$ holds.
- (c) L is projectable.
- (d) $\text{Reg}(L) = \text{Min}^*L = dL = \text{FP}(L) = \text{CP}(L)$.

Proof. Since every polar is an infimum of minimal primes, (a), evidently, implies (b). The equivalence of (b) and (c) is part of Theorem 3.1(a). Next, $\text{Reg}(L)$ is a subframe of L , which immediately shows that $\text{FP}(L) \subseteq \text{Reg}(L)$, if (b) holds. Then $\text{Reg}(L) = \text{Min}^*L = dL = \text{FP}(L) = \text{CP}(L)$ ensues. ■

The following corollary is an immediate consequence of the foregoing and 1.3(a).

Corollary 8.5. If PL is a sublattice of L then $\text{Reg}(L) = \text{Min}^*L = dL = \text{FP}(L)$.

Remark 8.6. Note that $\text{Reg}(L) = \text{Min}^*L$ implies that each minimal prime of L is regular. By applying Theorem 3.1(b), it is easily seen that the condition

every minimal prime of L is regular

is equivalent to $\text{Reg}(4)$.

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